A Mathematical Model for the Behavior of Pedestrians

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Abstract

The movement of pedestrians is supposed to show certain regularities which can be best described by an "algorithm" for the individual behavior and is easily simulated on computers. This behavior is assumed to be determined by an intended velocity, by several attractive and repulsive effects and by fluctuations. The movement of pedestrians is dependent on decisions, which have the purpose of optimizing their behavior and can be explicitly modelled. Some interesting applications of the model to real situations are given, especially to formation of groups, behavior in queues, avoidance of collisions and selection processes between behavioral alternatives.

Key words: pedestrians, movement, dynamics, motivation, conflicts, decisions, field theory, groups, queues, avoidance, territory, selection, break of symmetry

1 Introduction

Human behavior is based on individual decisions. In building a mathematical model for the movement of pedestrians, one has to assume that these decisions are not completely random, but show certain regularities instead. This assumption may be justified, because decisions and therefore the behavior of pedestrians will usually be determined by utility maximization: A pedestrian wants to move in a most convenient way, tries to minimize delays when having to avoid obstacles and other pedestrians, intends to take an optimal path and to walk with the minimal velocity allowing to reach a destination at a certain time, etc. The optimal behavior for a given situation can be derived by plausibility considerations and will be used as a model for pedestrian movement. Of course this optimal behavior is normally not thought about by an individual, but by trial and error it has automatically learned to use the most successful behavioral strategy, when being confronted with a standard situation (compare to sect. 3.2,(d)).

Due to several reasons we cannot expect the model to be *exactly* valid. Firstly an individual may find itself in a nonstandard situation. Secondly it probably has not learned the optimal strategy yet. Thirdly sometimes emotional or other reasons may lead to a

suboptimal behavior concerning its movement. Fourthly every behavior shows a certain degree of imperfection or irregularity. All these reasons lead to deviations from the optimal behavior and may be handled as fluctuations.

Nevertheless, the model gives a good impression of pedestrian movement: Firstly there is a tendency of pedestrians to move with an intended velocity (i.e. with an intended speed into an intended direction) (sect. 2.1). Secondly individuals sometimes like to approach or avoid certain objects or persons, which can be interpreted as attractive or repulsive effects (sect. 2.2). Especially, there is a necessity of avoiding the collision with obstacles and other pedestrians (sect. 2.2,(b)). The consequences of each aspect will be discussed in section 3 and can be compared directly with empirical observations. Some of them will be demonstrated by computer simulations (sect. 4).

2 The model

2.1 Intended velocity of motion

(a) If an individual i wants to arrive at a *destination* $\vec{x}_i^0$ at time T_i, being at time t at place $\vec{x}_i(t)$, its *ideal velocity* $\vec{u}_i^0(t)$ of movement will normally have the following properties (assuming a rectilinear way to the destination as easiest situation first):

- For convenience (in order to avoid deceleration and acceleration processes), the speed should be as uniform as possible, i.e.

$$u_i^0(t) \approx const.$$

- In walking the remaining distance

$$s_i(t) := \|\vec{x}_i^0 - \vec{x}_i(t)\|$$

  one should just use the remaining time $T_i - t$ (if one wants to avoid coming too late or too soon), i.e.

$$u_i^0(t) := \frac{s_i(t)}{T_i - t}\,.$$

- The direction $\vec{e}_i$ of moving should in the simplest case be *directly* oriented towards the destination $\vec{x}_i^0$, i.e.

$$\vec{e}_i := \frac{\vec{x}_i^0 - \vec{x}_i(t)}{\|\vec{x}_i^0 - \vec{x}_i(t)\|}\,.$$

All these properties are fulfilled by the ideal velocity

$$\vec{u}_i^0(t) = \frac{\vec{x}_i^0 - \vec{x}_i(t)}{T_i - t} = \frac{s_i(t)}{T_i - t}\vec{e}_i\,. \qquad (1)$$

Intending to move with velocity $\vec{u}_i^0(t)$ guarantees a uniform movement and, when suffering deviations or delays, an orientation towards the destination and an adaptation of speed. If the available way to the destination is not rectilinear, it can be

approximated by a polygon with edges $\vec{x}_i^n$, ..., $\vec{x}_i^0$, where $\vec{x}_i^n$ denotes the starting point. In that case, the formulas above remain unaltered, but the direction $\vec{e}_i := \vec{e}_i^j$ of movement is oriented towards the next edge $\vec{x}_i^j$, after having passed the edges $\vec{x}_i^n$, ..., $\vec{x}_i^{j+1}$:

$$\vec{e}_i^j := \frac{\vec{x}_i^j - \vec{x}_i(t)}{\|\vec{x}_i^j - \vec{x}_i(t)\|} .$$

Now we assume that an individual i of mass m_i, if moving with velocity $\vec{v}_i(t) := d\vec{x}_i(t)/dt$, applies a force

$$\vec{f}_i(t) \equiv m_i \frac{d\vec{v}_i(t)}{dt} := \gamma_i[\vec{v}_i^0(t) - \vec{v}_i(t)] \tag{2}$$

to get the acceleration $d\vec{v}_i(t)/dt$ towards the *intended velocity* of motion

$$\vec{v}_i^0(t) := \vec{e}_i \cdot \begin{cases} u_i^{min} & \text{for} \quad u_i^0(t) < u_i^{min} \\ u_i^0(t) & \text{for} \quad u_i^{min} \le u_i^0(t) \le u_i^{max} \\ u_i^{max} & \text{for} \quad u_i^0(t) > u_i^{max} . \end{cases} \tag{3}$$

According to this assumption, the force $\vec{f}_i$ is proportional to the discrepancy $\vec{v}_i^0 - \vec{v}_i$ between intended and actual velocity, and it vanishes, when both are equal ($\vec{v}_i = \vec{v}_i^0$). By (2) $\gamma_i\vec{v}_i(t)$ approaches $\gamma_i\vec{v}_i^0(t)$ exponentially with a relaxation time of m_i/γ_i. The quantity $\gamma_i\vec{v}_i^0$ has the meaning of the *motivation to get ahead* with velocity $\vec{v}_i^0$. For $\vec{v}_i^0$ we have introduced a cutoff at u_i^{max} and u_i^{min}, because velocities above u_i^{max} are felt strenuous or uncomfortable, and velocities less than u_i^{min} are felt "boring". u_i^{min} depends on the surroundings (see (d)). In the following we will assume the common case $\vec{v}_i^0 = \vec{u}_i^0$ (i.e. $u_i^{min} \le u_i^0 \le u_i^{max}$), if nothing contrary is mentioned.

There are some other types of pedestrian movement which can be formally reduced to type (a):

(b) Suppose that individual i has the *plan* to pass at times t through certain places $\vec{x}_i^0(t)$. Its intended velocity would then be

$$\vec{v}_i^0(t) = \frac{d\vec{x}_i^0(t)}{dt} .$$

But if the individual has, due to delays, at a certain time t_i still a distance $\Delta s_i(t_i) = \|\vec{x}_i^0(t_i) - \vec{x}_i(t_i)\|$ from its intended place $\vec{x}_i^0(t_i)$, it will try to make up for this distance during a time interval Δt_i, i.e. until time $t_i + \Delta t_i$. In that case, the intended velocity will, according to (1), be modified to

$$\begin{aligned} \vec{v}_i^0(t) &= \frac{\vec{x}_i^0(t_i + \Delta t_i) - \vec{x}_i(t)}{(t_i + \Delta t_i) - t} \\ &= \frac{\vec{x}_i^0(t_i + \Delta t_i) - \vec{x}_i^0(t)}{(t_i + \Delta t_i) - t} + \frac{\vec{x}_i^0(t) - \vec{x}_i(t)}{(t_i + \Delta t_i) - t} \\ &\approx \frac{d\vec{x}_i^0(t)}{dt} + \frac{\vec{x}_i^0(t) - \vec{x}_i(t)}{(t_i + \Delta t_i) - t} . \end{aligned}$$

(c) If an individual i intends to move with *constant velocity* v_i^0, we get type (1) by the identification

$$u_i^{max} := v_i^0 .$$

(d) Suppose individual i moves at leisure. Then it moves with a velocity

$$v_i^0(t) = u_i^{min}(\vec{x}_i(t)) \, ,$$

allowing to make as many interesting perceptions per time unit as intended. Therefore the appropriate velocity will depend on the actual place $\vec{x}_i(t)$. The intended direction $\vec{e}_i(t)$ of movement is given by spontaneous decisions (see section 2.2).

2.2 Contradictory motivations and decisions

An object or individual j sometimes induces a psychic reaction in a pedestrian i, motivating i to approach or avoid j [1]. These attractive or repulsive effects can be described by quantities $\vec{f}_{ij}^a$ or $\vec{f}_{ij}^r$ respectively, known as gradient of approach or avoidance. $\vec{f}_{ij}^{a/r}$ are not forces yet, but they are a measure for the direction and strength of the psychic motivation of i to approach or avoid j. The strength $f_{ij}^{a/r}$ of these motivations will lessen with increasing distance $r_{ij} = \|\vec{x}_j - \vec{x}_i\|$ of i and j, whereas the direction $\vec{e}_{ij}$ will be normally oriented towards or away from j , i.e.

$$\vec{e}_{ij} = \pm\widehat{\vec{r}_{ij}} = \pm\frac{\vec{r}_{ij}}{r_{ij}} := \pm\frac{\vec{x}_j - \vec{x}_i}{\|\vec{x}_j - \vec{x}_i\|}$$

(+: attractive case, –: repulsive case). So with

$$\vec{r}_{ij} := \vec{x}_j - \vec{x}_i = r_{ij} \cdot \widehat{\vec{r}_{ij}}$$

we find

$$\vec{f}_{ij}^{a/r}(\vec{r}_{ij}) = \pm f_{ij}^{a/r}(\vec{r}_{ij}) \cdot \widehat{\vec{r}_{ij}} \, . \tag{4}$$

In the absence of other motivations, the total effect

$$\vec{f}_{ij}(\vec{r}_{ij}) := \vec{f}_{ij}^a(\vec{r}_{ij}) + \vec{f}_{ij}^r(\vec{r}_{ij})$$

induced by j would play an analogous role as the motivation $\gamma_i\vec{v}_i^0$ to get ahead in equation (2) [2, 3]. This would lead to a movement according to

$$m_i\frac{d\vec{v}_i(t)}{dt} = \vec{f}_i(t) := \vec{f}_{ij}(\vec{r}_{ij}(t)) - \gamma_i\vec{v}_i(t) \, . \tag{5}$$

If individual i is subject to *a couple* of motivations, the total effect would be the sum of all, resulting in the following equation of motion generalizing (2) and (5):

$$m_i\frac{d\vec{v}_i(t)}{dt} = \vec{f}_i(t) := \left(\sum_j \vec{f}_{ij}(t) + \gamma_i\vec{v}_i^0(t)\right) - \gamma_i\vec{v}_i(t) \, . \tag{6}$$

But often it is not optimal to behave according to (6), namely in the case of contradictory motivations $\vec{f}_{ij}$, $\gamma_i\vec{v}_i^0$, which evoke a psychic *conflict*. Then it will be better for the individual to take a *decision*, whereby the behavioral alternative with the maximal *utility* will be prefered [4, 5]. In some cases this behavioral alternative can be a *compromise*. In other cases, namely when the alternatives in question mutually exclude each other, it

will correspond to the alternative which provides the *strongest* motivation. We now follow LEWINs "field theoretical" view [6]: Once a decision is taken, a *new* motivation

$$\vec{f}_i^0(t) \equiv \vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i \vec{v}_i^0(t), t\right)$$

arises as a substitute of the original motivations $\vec{f}_{ij}$, $\gamma_i \vec{v}_i^0$. This motivation is some kind of *psychic tension*, which causes the individual to act towards its aim in order to *diminish* this tension. In the case of pedestrians, the body will be induced to generate a *physical force*

$$\vec{f}_i(t) := \vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i \vec{v}_i^0(t), t\right) - \gamma_i \vec{v}_i(t)\,,$$

which then causes a movement according to

$$m_i \frac{d\vec{v}_i(t)}{dt} = \vec{f}_i(t) = \vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i \vec{v}_i^0(t), t\right) - \gamma_i \vec{v}_i(t) \tag{7}$$

(compare to (2), (6)). Due to (7), a pedestrian will stop moving only, when the motivation to move is vanishing, i.e. when

$$\vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i \vec{v}_i^0(t), t\right) = \vec{0}\,. \tag{8}$$

By

$$\vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i \vec{v}_i^0(t), t\right) := \sum_j \vec{f}_{ij} + \gamma_i \vec{v}_i^0$$

(6) can be interpreted as special case of (7), being valid as long as no decision is taken. In that case (8) has the form of an equilibrium condition for the motivations $\vec{f}_{ij}$, $\gamma_i \vec{v}_i^0$:

$$\sum_j \vec{f}_{ij}(t) + \gamma_i \vec{v}_i^0(t) = \vec{0}\,. \tag{9}$$

Now two examples for situations will be given, in which conflicts between several motivations occur:

(a) **Joining behavior**

Suppose individual i perceives an attractive object or individual j of attraction $f_{ij}^a(t_{ij})$ at time t_{ij}. Individual i will then spontaneously decide to meet j, if there is enough time to do so. We assume this to be the case if

$$f_{ij}^a(t_{ij}) > \gamma_i v_i^0(t_{ij}) = \gamma_i \frac{s_i(t_{ij})}{T_i - t_{ij}}\,,$$

i.e. if the motivation f_{ij}^a for joining j is greater than the motivation $\gamma_i v_i^0$ to continue walking (see (2)). (Here, we have made the simplification that there is only a small detour necessary to meet j.)

Individual i will stay at the meeting point for a time τ_{ij} and will leave at the moment $t_{ij} + \tau_{ij}$, when the tendency f_{ij}^a to join the attractive person or object j becomes less than the increasing tendency $\gamma_i v_i^0$ to get ahead. ($v_i^0(t)$ is growing according to the delay τ_{ij} resulting from the stay.) This condition can be written in the form

$$f_{ij}^a(t_{ij} + \tau_{ij}) \stackrel{!}{=} \gamma_i v_i^0(t_{ij} + \tau_{ij}) = \gamma_i \frac{s_i(t_{ij})}{T_i - (t_{ij} + \tau_{ij})} \tag{10}$$

(see (1)) because of $s_i(t_{ij}+\tau_{ij}) = s_i(t_{ij})$. By (10) the staying time τ_{ij} can be calculated as

$$\tau_{ij} = (T_i - t_{ij}) - \frac{\gamma_i s_i(t_{ij})}{f_{ij}^a} = (T_i - t_{ij}) \frac{f_{ij}^a - \gamma_i v_i^0(t_{ij})}{f_{ij}^a} \,, \tag{11}$$

if f_{ij}^a is constant with time ($f_{ij}^a(t) = f_{ij}^a$).

If $f_{ij}^a(t_{ij}) \le \gamma_i v_i^0(t_{ij})$ or, equivalently, $\tau_{ij} \le 0$, there is not enough time for joining j, and individual i will do best to continue walking without changing its way.

Summarizing (a), the decision of individual i leads to a new motivation

$$\vec{f}_i^0\left(\vec{f}_{ij}^a(t), \gamma_i \vec{v}_i^0(t)\right) := \gamma_i \vec{v}_i^0(t)\, \Theta\left(f_{ij}^a(t) < \gamma_i v_i^0(t)\right) \,,$$

which substitutes the contradictory motivations $\vec{f}_{ij}^a$ and $\gamma_i \vec{v}_i^0$. Here, we have introduced the decision function

$$\Theta(x) := \begin{cases} 1 & \text{if } x \text{ is true} \\ 0 & \text{if } x \text{ is false.} \end{cases}$$

Of course, individual i will change its direction of motion temporarily from $\vec{e}_i(t_{ij})$ to $\widehat{\vec{r}_{ij}}$, if this is necessary for joining j.

(b) **Avoidance behavior**

Suppose individual i, e.g. in order to avoid a collision, decides at time t_i to avoid an object or individual j (i.e. to keep a certain distance). Then, on one hand, individual i tries to minimize the maximal repulsive effect against j, namely

$$\max_t f_{ij}^r(\vec{r}_{ij}(t)) =: f_{ij}^r(\vec{r}_{ij}(t_{ij})) \,,$$

which normally occurs at the moment t_{ij} of greatest approach $r_{ij}(t_{ij})$. On the other hand, it wants to minimize the increase of the pressure $\gamma_i v_i^0$ to get ahead, i.e. to minimize the detour, which is necessary to avoid j. The best compromise will be to take a way, for which the maximal repulsive tendency and the tendency to get ahead have equal amounts, namely for which

$$f_{ij}^r(\vec{r}_{ij}(t_{ij})) = \gamma_i v_i^0(t_{ij}) \,, \tag{12}$$

and to take a rectilinear path. This path is given as tangent to the area

$$\mathcal{T}_{ij}(t) := \{\vec{x} : f_{ij}^r(\vec{x}_j(t) - \vec{x}) > \gamma_i v_i^0(t)\} \,, \tag{13}$$

which describes the *territory* of j, that is *respected*, i.e. not entered by individual i. Due to (13) the area of the respected territory $\mathcal{T}_{ij}(t)$ decreases with increasing intended velocity $v_i^0(t)$ or, equivalently, with increasing pressure $\gamma_i v_i^0(t)$ to get ahead.

For the sake of completion, we assume the following additional laws of pedestrian avoidance behavior:

- When avoiding a pedestrian or obstacle j, individual i will keep its intended speed $v_i^0(t_i)$, changing only its intended direction from $\vec{e}_i(t_i)$ to

$$\frac{\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)}{\|\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)\|},$$

where $\vec{x}_i^0(t_{ij})$ is the intended position of i for the moment of greatest approach. According to this, the motivation to get ahead will be changed from $\gamma_i \vec{v}_i^0(t_i)$ to

$$\vec{f}_i^0\left(\vec{f}_{ij}^r(t), \gamma_i \vec{v}_i^0(t)\right) := \gamma_i v_i^0(t) \frac{\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)}{\|\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)\|}$$

during the time it takes to avoid j (i.e. for times t with $t_i \leq t \leq t_{ij}$).

- An individual i reacts a time $\Delta t_{ij} := t_{ij} - t_i$ before a collision would be expected. This time Δt_{ij} is a psychic parameter, which, of course, will be the greater the larger the dimension of the obstacle j is. The *distance* d_{ij} of reaction before the location of a probable collision is the product of Δt_{ij} and speed v_i:

$$d_{ij} = v_i \Delta t_{ij}.$$

It is plausible, that the necessary angular change of direction when avoiding an obstacle j will be the greater, the lower the distance d_{ij} of the obstacle j is. So the (average) change of direction will be the greater, the lower the (average) speed is. This can be observed when comparing more and less crowded situations.

- If the distance for passing j on the left is nearly the same as for passing j on the right, we assume individual i to take the right hand side with probability p_1 and the left hand side with probability $p_2 := 1 - p_1$.

- But if there is no chance of passing j, e.g. when the way is too crowded, individual i will decelerate (as long as necessary) to a velocity $\vec{v}_i$, which allows a maximal component $\vec{v}_i \cdot \vec{e}_i$ of movement into the intended direction $\vec{e}_i$. This maximal component is normally equal to the component $\vec{v}_j \cdot \vec{e}_i$, which the hindering pedestrian's velocity $\vec{v}_j$ has in direction $\vec{e}_i$ (corresponding to the situation, that individual i walks in a gap behind a pedestrian j with velocity $\vec{v}_j$). However, if pedestrian j has an opposite direction with respect to i ($\vec{v}_j \cdot \vec{e}_i < 0$), it will be better for individual i to stop ($\vec{v}_i = \vec{0}$). Summarizing these results we have the relation

$$\vec{v}_i \cdot \vec{e}_i := \begin{cases} \vec{v}_j \cdot \vec{e}_i & \text{if } \vec{v}_j \cdot \vec{e}_i > 0 \\ 0 & \text{else.} \end{cases}$$

3 Conclusions and comparison with real situations

3.1 Effects of the intended velocity of motion

(a) **Velocity of motion**
According to (2) a pedestrian would normally walk with velocity $\vec{v}_i(t) \approx \vec{v}_i^0(\vec{x}_i(t))$.

But in order to avoid collisions, an individual i suffers detours or delays, and as a consequence, its *smoothed* velocity $\vec{\bar{v}}_i(t)$ of motion will probably have the more general form

$$\frac{d\vec{\bar{x}}_i(t)}{dt} = \vec{\bar{v}}_i(t) \approx \vec{w}_i + k_i \vec{v}_i^0(\vec{\bar{x}}_i(t)) = \vec{w}_i + k_i \frac{\vec{x}_i^0 - \vec{\bar{x}}_i(t)}{T_i - t} \tag{14}$$

with $k_i \le 1$ (see (1)). k_i and $\vec{w}_i$ are empiric parameters depending on the walking situation and describing the effect of "interindividual interactions". (14) can be solved by

$$\vec{\bar{v}}_i(t) = \vec{w}_i + \begin{cases} \left(\frac{k_i}{T_i - t_i^0}[\vec{x}_i^0 - \vec{\bar{x}}_i(t_i^0)] + \frac{k_i}{k_i - 1}\vec{w}_i \right) \left(1 - \frac{t - t_i^0}{T_i - t_i^0} \right)^{k_i - 1} - \frac{k_i}{k_i - 1}\vec{w}_i & \text{if} \quad k_i \ne 1 \\ \frac{1}{T_i - t_i^0}[\vec{x}_i^0 - \vec{\bar{x}}_i(t_i^0)] + \ln\left(1 - \frac{t - t_i^0}{T_i - t_i^0} \right) \cdot \vec{w}_i & \text{if} \quad k_i = 1 \,, \end{cases}$$

where t_i^0 is the time when individual i starts walking. We can conclude the following:

- If the smoothed actual velocity $\bar{v}_i$ is less than the intended velocity v_i^0, then $\bar{v}_i$ and v_i^0 will be growing with time, because from (14)

  $$\frac{d\vec{\bar{v}}_i(t)}{dt} \approx k_i \frac{\vec{v}_i^0(t) - \vec{\bar{v}}_i(t)}{T_i - t} = k_i \frac{d\vec{v}_i^0(t)}{dt}$$

  can be derived. So individual i will speed up in the course of time unless the maximal velocity $\vec{v}_i^{max} = \vec{w}_i + k_i u_i^{max} \vec{e}_i$ is reached. (Apart from (14) we have now taken into account eq. (3).)
- Individual i will arrive at destination $\vec{x}_i^0$ too late if the smoothed actual velocity $\bar{v}_i(t)$ would have to exceed the maximal velocity v_i^{max} before time T_i, i.e. if

  $$\lim_{t \to T_i} \bar{v}_i(t) > v_i^{max} \,.$$

- It will keep less distance to other pedestrians j as $\bar{v}_i(t)$ increases (see sect. 2.2,(b)), because of

  $$\gamma_i \vec{v}_i^0(t) = \gamma_i \frac{\vec{\bar{v}}_i(t) - \vec{w}_i}{k_i}$$

  and equation (12). Individual i then shows less respect against the "territory" of an individual j: it walks more aggressively and perhaps even pushes.
- In crowded situations individual i can prevent having to hurry by intending to walk with velocity

  $$\vec{v}_i^0 := -\frac{\vec{w}_i}{k_i} + \frac{1}{k_i} \frac{\vec{x}_i^0 - \vec{\bar{x}}_i(t)}{T_i - t} \,.$$

  This strategy will lead to a smoothed actual velocity of $\vec{\bar{v}}_i = \vec{v}_i^0$.

(b) **Effect of an unexpected detour**
In some situations an individual i has to walk an unexpected detour Δs_i, e.g. if it has forgotten something and is suddenly remembering this at time t_i^+. So the intended velocity changes according to (1) from

$$v_i^0(t_i^-) = \frac{s_i(t_i^-)}{T_i - t_i^-}$$

at the preceding moment t_i^- to

$$v_i^0(t_i^+) = \frac{s_i(t_i^+)}{T_i - t_i^+} = \frac{s_i(t_i^-) + \Delta s_i}{T_i - t_i^+} > v_i^0(t_i^-)\,.$$

By (2) this gives rise to a sudden increase of velocity v_i, which can often be observed, especially for individuals who walk according to a plan $\vec{x}_i^0(t)$ (see sect. 2.1,(b)). These individuals try to speed up to maximal velocity $v_i^0(t_i^+) := u_i^{max}$ until they have, after a time interval

$$\Delta t_i \geq \frac{\Delta s_i}{u_i^{max} - v_i^0(t_i^-)}\,,$$

reached their plan $\vec{x}_i^0(t_i + \Delta t_i)$ again (in the sense of $\vec{x}_i(t_i + \Delta t_i) = \vec{x}_i^0(t_i + \Delta t_i)$).

(c) **Behavior in a queue**

If the front of a queue has come to rest, the following phenomenon can often be observed: After a while, one of the waiting individuals begins to move forward a little, causing the successors to do the same. This process propagates in a wave-like manner to the end of the queue, and the distance to move forward increases.

Why do individuals behave in such a paradox way?—They don't get away any faster but only cause the queue to become more crowded! Our model gives the following interpretation:

At time t_i an individual i keeps a distance $r_{i,i-1}(t_i)$ to the individual $i-1$ in front, which is (according to (9) and (12)) given by

$$f_{i,i-1}^r(r_{i,i-1}(t_i)) = \gamma_i v_i^0(t_i)\,.$$

$f_{i,i-1}^r$ is the repulsive effect describing the territory of individual $i-1$ respected by i. As we know from (1), $v_i^0(t)$ grows as time t passes, because individual i is at rest ($\vec{x}_i(t) = \vec{x}_i(t_i)$). So at time $t_i + \Delta t_i$ individual i would prefer to have a distance

$$r_{i,i-1}(t_i + \Delta t_i) =: r_{i,i-1}(t_i) - \Delta r_i(t_i + \Delta t_i)\,,$$

which has reduced by an amount Δr_i and is given by

$$f_{i,i-1}^r(r_{i,i-1}(t_i + \Delta t_i)) = \gamma_i v_i^0(t_i + \Delta t_i)\,. \tag{15}$$

But individual i moves up a distance Δr_i only if

$$\Delta r_i \geq \Delta r_i^{min}\,, \tag{16}$$

i.e. if the increment Δr_i exceeds a minimal stride Δr_i^{min}. So the first individual moving up is the individual i, for which condition

$$\Delta r_i(t_i + \Delta t_i) = \Delta r_i^{min}$$

is fulfilled first. This is the case at a time $t := t_i + \Delta t_i$, i.e. a time interval Δt_i after its last step at time t_i. Now the successors $i+n$ ($n \geq 1$) will move forward a distance

$$s_{i+n} = \sum_{j=i}^{i+n} \Delta r_j(t) = \sum_{j=i}^{i+n} \Delta r_j(t_j + \Delta t_j)$$

according to (15) and (16), because $s_{i+n} \geq \Delta r_{i+n}^{min}$ will normally be fulfilled.

3.2 Attractive and repulsive effects

(a) **Constant density**

Suppose a number of N individuals having only a negligible intention to move ($\vec{v}_i^0 \approx \vec{0}$) stay in an area of a (dining) hall, a waiting room, a beach, an underground station, etc. with size A. One can observe then a quite uniform distribution of individuals (with constant density N/A) if there are no special attractions in area A and no aquaintances between the individuals (see (b)). This is due to the repulsive effects $\vec{f}_{ij}^r$ between each pair of individuals i and j, which are in equilibrium (see (9)), when all individuals occupy a personal territory of nearly equal size.

(b) **Formation of groups**

If there *are* aquaintances between the individuals of example (a), a truncated Poisson distribution

$$p_k = \mathcal{N} \frac{\lambda^k}{k!} \qquad k = 1, 2, \dots \tag{17}$$

can be found for the proportion p_k of groups consisting of k members. This distribution is well confirmed by empirical data [7] and can be explained by the following mathematical model of Coleman [8, 7]:

$$\begin{aligned}
\frac{dp_k}{dt} &= [\text{transitions from } l(\neq k) \text{ to } k \; - \; \text{transitions from } k \text{ to } l(\neq k)]/\text{time unit} \\
&= \sum_{l(\neq k)} p_l \cdot r(l \to k) - \sum_{l(\neq k)} p_k \cdot r(k \to l) \\
&= (p_{k+1} \cdot (k+1) \cdot \beta + p_{k-1} \cdot \alpha \cdot p_1) - (p_k \cdot k \cdot \beta + p_k \cdot \alpha \cdot p_1)
\end{aligned} \tag{18}$$

for $k = 2, 3, \dots$, and

$$\sum_{k=1}^{\infty} p_k = 1\,. \tag{19}$$

In (18) we have used

$$r(k \to l) = \begin{cases} k \cdot \beta & \text{if } l = k-1 \\ \alpha \cdot p_1 & \text{if } l = k+1 \\ 0 & \text{else} \end{cases}$$

with $l \geq 2$. This means that a group with k individuals loses individuals independently with rate β and gains single individuals with rate $\alpha \cdot p_1$ (which is proportional to the number of single individuals). Other transitions are assumed to be relatively unimportant.

(18), (19) have the stationary solution

$$p_k = \frac{1}{\mathrm{e}^\lambda - 1} \frac{\lambda^k}{k!}\,,$$

given by $dp_k/dt = 0$, where

$$\lambda := \ln\left(\frac{\alpha}{\beta} + 1\right)\,. \tag{20}$$

We now connect these results with our model: For β we could simply take the mean value of the reciprocal $1/\tau_{ij}$ of the time τ_{ij} which an individual i stays in a group j, because this is the rate of leaving a group (see (11)):

$$\beta := E\left(\frac{1}{\tau_{ij}}\right) . \tag{21}$$

On the other hand, α can be assumed of the form

$$\alpha := p_+ J \, , \tag{22}$$

where J is the rate of recognized groups per time unit and p_+ is the probability to *join* a recognized group j. According to sect. 2.2,(a), p_+ is the probability $P(\tau_{ij} > 0)$, that the staying time τ_{ij} is positive:

$$p_+ := P(\tau_{ij} > 0) = P(f^a_{ij} > \gamma_i v^0_i) . \tag{23}$$

f^a_{ij} is, of course, the attractive effect between individual i and group j.

Due to (20) to (23) the following conclusions can now be made:

- Parameter λ, which is a measure for the average number of members of a group, increases with the mean value of the staying time τ_{ij}, i.e. it decreases with growing intended velocity v^0_i and increases with growing remaining time $T_i - t_{ij}$ (see (11)). This is consistent with the data [7].
- If the motivation f^a_{ij} to join a group j is less than the motivation $\gamma_i v^0_i$ to get ahead for all individuals i and groups j, we have $p_+ = 0$ and $\alpha = 0$. In that case no groups are forming at all and (a) can be applied again (if $\vec{v}^0_i \approx \vec{0}$).

(c) **Superposition of attractive and repulsive effects**
Often a person or object j has an attractive effect $\vec{f}^a_{ij}$ and a repulsive effect $\vec{f}^r_{ij}$ as well. As a consequence of equation (5), individual i will then show one of several characteristic dynamic behaviors known from approach-avoidance conflicts, depending on the special form of the motivation gradient $\vec{f}_{ij}(\vec{r}_{ij})$ [9]. Especially, for negligible intention to move ($\vec{v}^0_i \approx \vec{0}$), individual i will prefer a certain distance [2, 10], for which the equilibrium condition

$$f_{ij}(\vec{r}_{ij}) = f^a_{ij}(\vec{r}_{ij}) - f^r_{ij}(\vec{r}_{ij}) = 0$$

is fulfilled (see (9) and (4)), i.e. for which the attractive and the repulsive effect have equal strengths.

(d) **Break of symmetry for avoidance behavior**
Suppose two individuals walk in opposite direction and try to avoid each other in order not to suffer a collision. Then each tries to pass the other with probability p_1 on the right and probability $p_2 = 1 - p_1$ on the left (see sect. 2.2,(b)).

The probability for avoiding each other successfully is then

$$p_1 \cdot p_1 + p_2 \cdot p_2 =: 1 - w \, .$$

Otherwise, with probability

$$w = p_1 \cdot p_2 + p_2 \cdot p_1 = 2p_1 \cdot p_2 \leq \frac{1}{2}\,, \tag{24}$$

they have to try again, etc., until they pass on *different* sides. This phenomenon is well known.

The mean value $E(n)$ for the necessary number n of attempts to avoid each other is given by

$$E(n) = \sum_{n=1}^{\infty} n \cdot w^{n-1} \cdot (1-w) = \frac{1}{1-w}\,. \tag{25}$$

Taking (24) into account, this expression is *maximal* for $w = 1/2$, i.e. for symmetric probabilities

$$p_1 = p_2 = \frac{1}{2}$$

of avoidance for both sides. (25) is *minimal* for $p_1 = 0$ or $p_1 = 1$ (deterministic behavior!). Therefore *asymmetric* probabilities $p_1 \neq p_2$ of avoidance are favourable. In fact, in most countries individuals more frequently pass other individuals on the right ($p_1 > 1/2$). As a consequence, crowded ways often show two different lanes of opposite direction, which stick to the right side respectively [11, 12, 13]. This behavior reduces the frequency of situations of avoidance and corresponding delays.

Selection of one behavioral alternative

For explanation of the break of symmetry ($p_1 \neq p_2$), we consider the following general model which describes the temporal change of the proportion p_k of individuals showing a certain behavioral alternative k (compare to [14]):

$$\begin{aligned} \frac{dp_k}{dt} &= \sum_{l(\neq k)} (M_{kl}p_l - M_{lk}p_k) \\ &+ s_k p_k + \xi_k\,. \end{aligned} \tag{26}$$

M_{kl} are the *mutation* rates for changes from behavior l to behavior k per time unit and person. For the choice

$$s_k := M_{kk} - \sum_l M_{ll} p_l\,, \tag{27}$$

$s_k p_k$ has the effect of a *selection* between the behavioral alternatives k. ξ_k are random fluctuations of the proportion p_k.

For the problem of avoidance we have only two alternatives: one to pass a hindering pedestrian on the right ($k := 1$), and the other to pass it on the left ($k := 2$). As mutation matrix we take

$$\underline{M} := \underline{A} + \underline{B} \tag{28}$$

with

$$\underline{A} := \lambda \begin{pmatrix} p_1 & 1-p_2 \\ 1-p_1 & p_2 \end{pmatrix} \tag{29}$$

and

$$\underline{B} := \beta \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\,. \tag{30}$$

According to $\underline{A}$, a behavioral alternative k becomes more probable (by learning), the greater the proportion p_k of individuals with behavior k is (because in our case behavior k is the more *successful* the more often it occurs) [15, 16]. On the other hand, $\underline{B}$ describes a random choice of some behavior k with probability 1/2 due to trial (and error). (The individual behavior depends on the respective situation.)

Substitution of (27) to (30) in (26) now gives

$$\frac{dp_k}{dt} = [2\lambda p_k \cdot (1 - p_k) - \beta] \cdot \left(p_k - \frac{1}{2}\right) + \xi_k \, , \tag{31}$$

which, for $\beta \geq \lambda/2$, has the only stationary solution $p_k = 1/2$. However, for a low tendency β to choose the behavior randomly ($0 \leq \beta < \lambda/2$), (31) has three stationary solutions: $p_k = 1/2$, being *unstable* against fluctuations ξ_k, and $p_k = 1/2 \cdot (1 \pm \sqrt{1 - 2\beta/\lambda})$, being *stable*! As a consequence of the instability of $p_k = 1/2$, fluctuations will cause the proportion p_k to tend either towards $p_k = 1/2 + 1/2\sqrt{1 - 2\beta/\lambda}$ (prefering the right side) or towards $p_k = 1/2 - 1/2\sqrt{1 - 2\beta/\lambda}$ (prefering the left side). By spatial diffusion of this learning process the prefered behavior is spread over wide areas (e.g. countries) and stabilized against crossing $p_k = 1/2$, which could in principal be induced by fluctuations.

We now assume that an individual i *overtakes* a pedestrian j walking in the *same* direction. Here, we normally do not have to expect any complications by the behavior of j. So the avoidance behavior will be successful with probability $w = 1$, regardless of the side of passing. Our mutation matrix $\underline{M}$ then will not depend on the proportions p_1, p_2 of pedestrians passing on the left or on the right ($\lambda = 0$). This time we have the equation

$$\frac{dp_k}{dt} = -\beta \left(p_k - \frac{1}{2}\right) + \xi_k$$

(see (31)), which has only one stationary solution: the symmetric probability $p_k = 1/2$ of avoidance, which is stable!

4 Computer simulations

In order to test the somewhat algorithmical model of section 2 (especially section 2.2,(b)), some simple computer simulations have been carried out. The corresponding computer program works as follows:

- First the geometrical configuration is determinded (e.g. a normal pedestrian way or a pedestrian way with several obstacles).

- In the examples presented, two types (i.e. main directions) of motion are necessary: Pedestrians intending to walk from the left to the right are represented by black lines, those intending to walk in the opposite direction are represented by grey lines. Every line has the meaning of an individual's actual stride, and its length is proportional to its velocity.

- As initial configuration a statistically uniform spatial distribution of N pedestrians is taken ($N = 350$ or 500), one half belonging to the black type of motion, the other half belonging to the grey type (see fig. 1). The intended speeds of each direction are distributed by chance (GAUSSian), whereby the same mean speeds and the same velocity variances were chosen for both directions of motion.

- At the beginning of the simulation, a certain order of the N pedestrians is chosen at random. The pedestrians take each step according to that order. After even the Nth pedestrian has taken its Sth step, the 1st pedestrian is taking its $(S+1)$st one. For each individual leaving on one side of a figure, an equivalent one enters on the other side, i.e. the right side of each figure can be assumed to be connected to the left side (periodic boundary conditions).

- Now the considerations from section 2.2,(b) are taken into account: A pedestrian taking its next step will move by its intended stride into its intended direction, if this is possible. If not, i.e. if it would have to cross another pedestrian's step, it will change its direction by an angle, which will be the greater, the nearer the hindering pedestrian is. However, if even this does not prevent him from crossing another pedestrian's step, the intended stride will be taken as short as necessary, possibly leading to a stop. In the case of a change of direction, the right side is chosen with probability

$$p_1 := \begin{cases} 1/2 & \text{if both pedestrians belong to the same direction of motion} \\ p & \text{if the pedestrians belong to different directions of motion}\,. \end{cases}$$

  The left side is chosen with probability $p_2 = 1 - p_1$.

- If a pedestrian comes into the proximity of an obstacle, it temporarily changes its intended direction. It prefers to pass the obstacle at the nearest side in order to suffer the least possible detour. If both sides have approximately the same distance, each side is chosen with probability 1/2.

The computer simulations show the following results:

- For symmetric avoidance behavior ($p = 1/2$), changes of direction appear very often, because encounters of pedestians from opposite directions are likely to happen everywhere (see fig. 2). In the case of *asymmetric* avoidance behavior ($p = 0.7$), two walking lanes of opposite direction develop in the course of time (see fig. 3). Obviously, there are less changes of direction necessary, occuring mainly at the borderline between the opposite lanes.

- In the presence of an obstacle, a pedestrian free area develops in front of and behind the obstacle (see figures 4 and 5). But, whereas an obstacle in the middle of a pedestrian way causes only a small area not to be used (see fig. 4), obstacles at the margin do reduce the effective width over a long distance (see fig. 5).

5 Conclusions

We have set up a model for the movement of pedestrians starting from the idea that individual decisions are guided by maximization of utility. Once a decision is taken, a special kind of psychic motivation or tension to realize this decision arises, which causes the individual to act towards its aim in order to neutralize the psychic tension. For example, when an individual i wants to reach a certain destination at a time T_i, it would do best to walk with a suitable velocity $\vec{v}_i^0$. So the pedestrian will decide to walk with the "intended velocity" $\vec{v}_i^0$, causing it to apply a physical force $\vec{f}_i$, which vanishes, when the pedestrian's actual velocity $\vec{v}_i$ is equal to the intended one. In the case of delays, the intended velocity has to be corrected upwards in the course of time, causing the pedestrian to speed up and perhaps to walk more aggressively. Waiting in a queue that has come to rest, an individual will instead move forward after some time, which is motivating the successors to move forward, too. Therefore, this behavior propagates in a wave-like manner to the end of the queue and leads to a more crowded queue.

In addition, a pedestrian is subject to attractive or repulsive influences, motivating it to approach or to avoid certain individuals or things j. If, for example, the motivation $\vec{f}_{ij}^a$ to approach some person (say a friend) or some object (e.g. a shop-window) is greater than the motivation to get ahead, the pedestrian i will decide to join this individual or object for a while. But it will leave the moment at which the motivation to join the attractive person or object j becomes less than the increasing motivation to get ahead with the intended velocity (which is growing according to the delay resulting from the stay). If, right from the beginning, the motivation of a pedestrian to get ahead is greater than the motivation to join a certain person or object j, the pedestrian's best decision will be not to change its path at all. This model leads to a detailed description of group formation.

However, there are also repulsive effects $\vec{f}_{ij}^r$. They describe, for example, the personal territories of individuals j. As a consequence, individuals who don't know each other normally spread uniformly in an area of a hall, a waiting room, a cafe, a beach, etc. (if there are no special attractions). In situations where a pedestrian i has to avoid another one j in order to prevent a collision, it prefers to suffer only a minimal detour. So individual i will pass individual j along a tangent to the territory of j respected by i. This respected territory is given as the area around j, for which the repulsive effect f_{ij}^r of j is greater than the motivation $\gamma_i v_i^0$ of i to get ahead with speed v_i^0.

Mathematically, it appears to be favourable when most pedestrians prefer either the right side or the left side when passing each other. This results in the development of walking lanes in pedestrian crowds. With both sides being equivalent, one side will be used by a growing majority, once it has been chosen at random. This is one example being representative for many others, where the most successful or most efficient behavior is adopted by trial and error causing a selection between behavioral alternatives.

After having set up a "microscopic" model, i.e. one for the movement of *individuals*, one may be interested in a model for a great number of interacting pedestrians. Such a model is developed in [17]. It shows some similarities to gaskinetic and fluid dynamic equations, but contains some additional terms that are characteristic for pedestrian movement.

Acknowledgements:
First I want to thank Prof. Dr. M. R. Schroeder to give me the chance to work on

an interdisciplinary field: modelling the *social* behavior of pedestrians by *mathematical* models. Secondly I am grateful to Prof. Dr. R. Kree, Prof. Dr. W. Scholl, Prof. Dr. W. Weidlich, Dr. habil. G. Haag and Dr. R. Reiner for their stimulating discussions. Last but not least I'm obliged to D. Weinmann, N. Empacher, K.-G. Haas and E. Webster for reading, correcting and commenting my manuscripts.

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Fig. 1 ($N = 500$, $S = 0$):
Initial configuration: N pedestrians with varying speeds are distributed randomly over a pedestrian way, the black ones walking from left to right, the grey ones walking in opposite direction.

Fig. 2 ($N = 500$, $S = 500$, $p = 1/2$):
In order to avoid collisions with other pedestrians the direction of walking has to be changed often.

Fig. 3 ($N = 500$, $S = 500$, $p = 0.7$):
If the probability p for passing a hindering pedestrian on the right is different from the probability $1 - p$ for passing it on the left, two lanes of opposite direction develop.

Fig. 4 ($N = 350$, $S = 540$, $p = 0.7$):
In front of and behind an obstacle a pedestrian free area develops.

Fig. 5 ($N = 350$, $S = 540$, $p = 0.7$):
Obstacles at the margin of a pedestrian way reduce its effective width.

A Mathematical Model for the Behavior of Pedestrians

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Abstract

The movement of pedestrians is supposed to show certain regularities which can be best described by an "algorithm" for the individual behavior and is easily simulated on computers. This behavior is assumed to be determined by an intended velocity, by several attractive and repulsive effects and by fluctuations. The movement of pedestrians is dependent on decisions, which have the purpose of optimizing their behavior and can be explicitly modelled. Some interesting applications of the model to real situations are given, especially to formation of groups, behavior in queues, avoidance of collisions and selection processes between behavioral alternatives.